Geostrophic Balance

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Contents

1 Introduction

Here we'll introduce the concept of scaling, which is a formal way to estimate the magnitude of terms in an equation. Scaling is essentially a rule of thumb to decide which terms are dominant and which are negligible.

1.1 Notation

In what follows I'll assume an Eulerian frame of reference. First a quick bit of notation that I'll use throughout.

Vectors will be defined using bold text:

$$\mathbf{x} = (x,y,z)$$

Similarly for vectors of velocities:

$$\mathbf{U} = (u,v,w) = \frac{d\mathbf{x}}{dt}$$

And the horizontal component of a velocity vector: :

$$\mathbf{u} = (u,v)$$

Material derivatives:

$$\frac{Dc}{Dt} = \frac{\partial c}{\partial t} + \mathbf{U} \cdot \nabla c$$

When we want only the horizontal gradient, we will use ∇_z

1.2 Definition of terms

As in our notes on the primitive equations, we'll keep track of the definitions of variables in this section for reference later.

$$\Phi = gz \qquad \text{(geopotential height)}$$

$$b = -g\frac{\rho'}{\rho_0} = -g\frac{T'}{T_0} \qquad \text{(buoyancy)}$$

$$\phi = \frac{p'}{\rho} \qquad \text{(dynamic pressure)}$$

$$f \approx f_0 + (\beta y) = f_0 + \left(\frac{\partial f}{\partial y}\right)y = f_0 + \left(\frac{2\Omega cos\theta_0}{a}\right)y \qquad \text{(Coriolis parameter)}$$

2 Scale analysis

2.1 Hydrostatic balance

In what follows we will replace exact variables instead with estimates of their magnitude. We will start with the vertical component of the fully expanded equation for motion in a beta-plane prior to excluding any terms:

$$\frac{\partial w}{\partial t} + u\frac{dw}{dx} + v\frac{dw}{dy} + \frac{1}{2}\frac{dw^2}{dz} - 2(\Omega^y u) = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

which, in a scale analysis becomes:

$$\frac{W}{T} + 2U\frac{W}{L} + \frac{1}{2}\frac{W^2}{H} - 2(\Omega U) \sim -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

but because we care only about the magnitude of the terms

$$\frac{W}{T} + \frac{UW}{L} + \frac{W^2}{H} - \Omega U \sim -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

where W is the scale term for vertical velocity, U is the scale term for horizontal velocity, H is the scale term for height, L is the scale term for length and T is the scale term for time. Note that here we have not scaled the right hand side. Generally all of the terms on the left hand side are sufficiently small such that the right hand side dominates. To see this plug in magnitudes for each term yourself. Se the below table for typical horizontal values, and W and H are an order of magnitude smaller than their horizontal counterparts (CHECK THIS).

Variable	Scaling symbol	Meaning	Atmos. value	Ocean value
(x, y)	L	Horizontal length scale	10^6 m	10^5 m
t	T	Time scale	1 day (10^5 s)	10 days (10^6 s)
(u, v)	U	Horizontal velocity	$10\,\mathrm{m\,s^{-1}}$	$0.1\,\mathrm{m\,s^{-1}}$
	Ro	Rossby number, U/fL	0.1	0.01

Figure 1: Table 2.1 from Vallis (2006)

If the right hand side dominates, then we are left only with the hydrostatic balance!

$$-\frac{\partial p}{\partial z} = \rho g$$

2.2 Geostrophic balance

Having conducted a scale analysis on the vertical component of the momentum equation, we will now move to the horizontal components. We will find that the dominant terms in the horizontal direction are the pressure gradient terms and the Coriolis parameter. This balance is known as *geostrophic balance*. In order for geostrophic balance to hold, the Rossby number – which is a ratio of the magnitude of the advective and Coriolis terms – must be small. We can see where the Rossby number comes from by starting with the horizontal components of the momentum equation.

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{U} \cdot \nabla)\mathbf{u} + \mathbf{f} \times \mathbf{u} = -\frac{1}{\rho}\nabla_z p$$

$$\frac{U}{T} + \frac{U^2}{L} + fU \sim -\frac{1}{\rho}\nabla_z p$$

Now we take the ratio of the relative and Coriolis accelerations to define the Rossby number

$$R_0 \equiv \frac{\frac{U^2}{L}}{fU} = \frac{U}{fL}$$

If the Rossby number is small, then the (Coriolis) rotation term will dominate the (nonlinear) advective term, and we will be left with only

$$\mathbf{f} \times \mathbf{u} \approx -\frac{1}{\rho}\nabla_z p$$

which is the geostrophic balance! In cartesian coordinates:

$$fu \approx -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{1}$$

$$fv \approx \frac{1}{\rho}\frac{\partial p}{\partial x} \tag{2}$$

Notice that there is no vertical component of the geostrophic flow, which means that geostrophic flow is parallel to lines of constant pressure. In the northern hemisphere (where f is positive), geostrophic flow is anticlockwise around low pressure and clockwise around high pressure (see below).

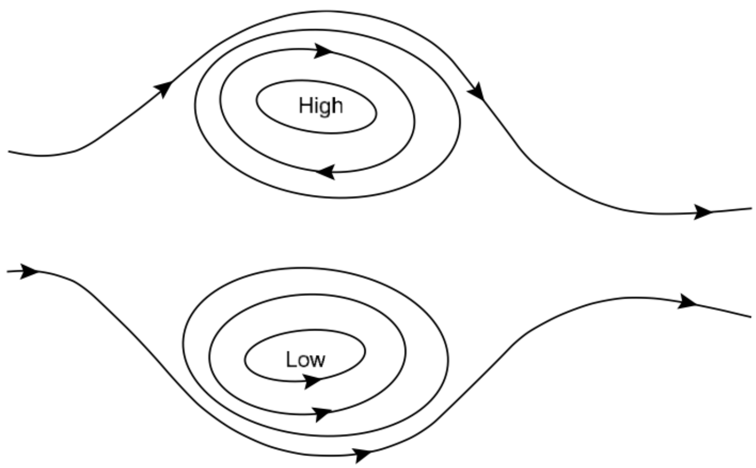

Fig. 2.5 Schematic of geostrophic flow with a positive value of the Coriolis parameter f. Flow is parallel to the lines of constant pressure (isobars). Cyclonic flow is anticlockwise around a low pressure region and anticyclonic flow is clockwise around a high. If f were negative, as in the Southern Hemisphere, (anti)cyclonic flow would be (anti)clockwise.

Figure 2: Figure 2.5 from Vallis (2006)

We will also note that if the density and the Coriolis force are constant in the horizontal, then the geostrophic flow is non-divergent ($\nabla_z \cdot \mathbf{u}_g = 0$)

2.2.1 Taylor-Proudman effect

2.3 Thermal wind

We may gain further insight by combining the geostrophic and hydrostatic approximations. This will be most straightforward in pressure coordinates. The hydrostatic balance is

$$\frac{\partial \mathbf{\Phi}}{\partial p} = -\frac{1}{\rho}$$

and if we substitute the ideal gas law ($\rho = p/RT$) into the hydrostatic balance

$$\frac{\partial \mathbf{\Phi}}{\partial p} = -\frac{RT}{p}$$

In pressure coordinates the geostrophic balance is

$$\mathbf{f} \times \mathbf{u}_g = -\nabla_p \mathbf{\Phi}$$

where ∇_p is the gradient operator applied on surfaces of constant pressure. If we differentiate the geostrphic balance with respect to pressure, we get:

$$\mathbf{f} \times \frac{\partial \mathbf{u}_g}{\partial p} = -\nabla_p \frac{\partial \mathbf{\Phi}}{\partial p}$$

and substituting the hydrostatic balance into the right hand side:

$$\mathbf{f} \times \frac{\partial \mathbf{u}_g}{\partial p} = -\nabla_p \left(-\frac{RT}{p} \right) = \frac{R}{p} \nabla_p T$$

Which is the thermal wind balance! In component form, it is

$$-f \frac{\partial v_g}{\partial p} = \frac{R}{p} \frac{\partial T}{\partial x} \tag{3}$$

$$f \frac{\partial u_g}{\partial p} = \frac{R}{p} \frac{\partial T}{\partial y} \tag{4}$$

These equations imply that changes in the geostrophic velocity with height are related to horizontal changes in temperature. In other words, a negative equator-to-pole temperature gradient may be responsible for strengthening the geostrophic wind with height. Note that these equations are in terms of pressure. Switching to z coordinates would change the sign of the left hand side since pressure increases downward.

2.4 Zonal winds

We can predict a great deal about the zonal flow using only the thermal wind balance and the observed equator-to-pole temperature gradients that we discussed in the previous section. Remember that the thermal wind balance is

$$f \frac{\partial u}{\partial p} = \frac{R}{p} \frac{\partial T}{\partial y}$$

which relates the vertical wind shear to meridional gradients in temperature. To get a full picture of the zonal wind, however, we should make a few additional notes about the structure of meridional temperature gradients throughout the atmosphere. In the troposphere, temperature falls monotonically with latitude. This gradient is larger in the winter hemisphere than the summer because peak insolation is shifted towards the summer hemisphere, so the winter pole receives virtually no direct sunlight. Above the troposphere (starting somewhere between 8 and 16 km above the sea level) is the stratosphere, where temperature increases with height. In the stratosphere the meridional temperature gradient is reversed (or, for the upper stratosphere, monotonic from pole-to-pole).

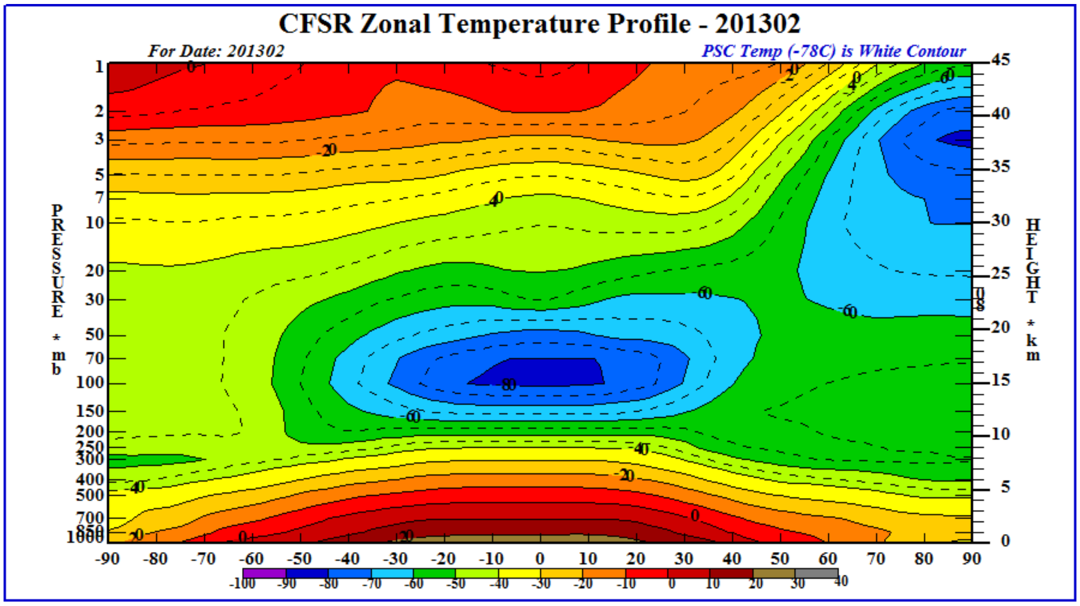

Figure 3: Stratospheric and tropospheric zonal temperature profile during northern hemisphere winter (taken in February).
Credit: National Weather Service, Climate Prediction Center website `www.cpc.ncep.noaa.gov/products/stratosphere/theta/theta_info.shtml`

By mentally applying the thermal wind relation to the above figure and comparing it to observations of zonal wind in the below figure, we can see the qualitative resemblance (note that the zonal wind figure reaches only 30 km while the temperature figure reaches 45 km). In the troposphere, meridional temperature gradients are largest at the edge of the subtropics, which leads to zonal jets, with the winter hemisphere jet being stronger than the summer hemisphere jet. The zonal winds also follow expectations in the stratosphere.

At the surface, winds alternate from easterly near the equator to westerly in the midlatitudes and finally easterly again at high latitudes. Surface winds are stronger in the southern hemisphere due to a lack of drag from continental land masses. And while only one season is shown here, surface winds within a given hemisphere are stronger in the winter as compared to the summer season (corresponding to the stronger jet aloft).

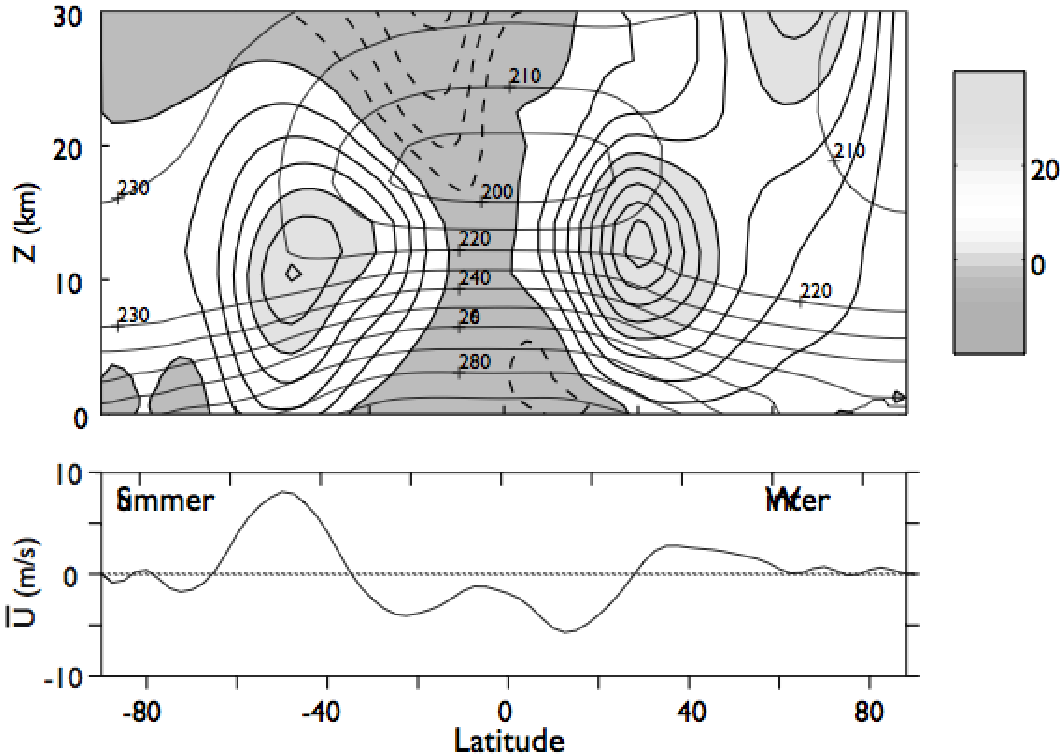

Figure 4: Top panel: zonally averaged zonal wind, bottom panel: zonally averaged zonal winds at the surface
Credit: Vallis (2006)

A significant result of these observations is that there is no need to invoke a convergence of momentum to drive these jets. They may result from the temperature gradient and thermal wind alone. However, just as the observed meridional temperature gradient is smaller than predicted by radiative convective equilibrium, by extension the zonal wind shear is also smaller than what would be predicted by radiative convective equilibrium. The net impact of large scale circulations, particularly turbulent motions in the mid-latitudes, is to reduce the amplitude of temperature and zonal wind shear via poleward energy transport.

References

Geoffrey K Vallis. *Atmospheric and oceanic fluid dynamics: fundamentals and large-scale circulation.* Cambridge University Press, 2006.